The Mathematics Of Stock Option Valuation - Part Six
Solution Via Finite Difference Approximations

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We have a call option on an underlying stock that can be exercised at time $t = T$ where $T > 0$. Because at time T we know the value of the underlying stock we also know the value of the call option, which is...

$$C_T = Max(S_T - K, 0) \tag{1}$$

Note that C_T is the value of the call option at time T, S_T is the price of the underlying stock at time T and K is the exercise price.

At time $t = 0$ we do not know the value of the underlying stock at time $t = T$ and therefore we do not know the value of the call option at time $t = T$. Even though from the vantage point of time $t = 0$ we do not know the value of the call option at time T we do have an idea as to the range of possible values. Our goal is to use a Finite Difference Method to approximate the value of the call option at time $t = 0$. Note that for ease of exposition we will be using the Explict Euler Method, which is conditionally unstable and whereas its the easiest of the finite difference methods to implement and explain it should not be used for serious derivative valuation work.

Our Hypothetical Problem

Assume that we have a call option with the following parameters...

Stock price at time zero	S_0	\$60.00
Call option exercise price	K	\$60.00
Annual return volatility	σ	0.20
Annual risk-free rate	r	0.05
Time to option expiration in years	T	1.00

Problem: Determine the value of C_0, which is the value of the call option at time $t = 0$?

The Finite Difference Approximation

We will divide the time interval [0,T] into M equally sized subintervals of length Δt such that...

$$\Delta t = \frac{T}{M} \;\; ...where... \;\; m = \{0, 1, 2, ..., M\} \tag{2}$$

We will define S_{max} to be the maximum stock price that can be obtained by time T. We will divide S_{max} into N equally sized subintervals of length Δs such that...

$$\Delta s = \frac{S_{max}}{N} \;\; ...where... \;\; n = \{1, 2, ..., N\} \tag{3}$$

The space interval, which is the range of possible stock prices, is therefore $[\Delta s,\, N\,\Delta s]$. Using equation (1) above, the equation for the payoff on the call option at time T as a function of n, which is unknown at time $t = 0$, is...

$$C_T = Max(n\,\Delta s - K, 0) \tag{4}$$

We will create a rectangular grid with N rows and M + 1 columns. The horizontal axis will represent the time interval, which goes from $m = 0$ to $m = M$, and the vertical axis will represent the space interval, which goes from $n = 1$ to $n = N$. Our task is to discretize the Black-Scholes partial differential equation (PDE) and work it backwards in time from time $m = M$ to time $m = 0$. We work backwards in time because we know the possible call option payoffs (i.e. call value) at time $m = M$ and its the value at time $m = 0$ that we seek. We will represent the value of the call option as $f_{m,n}$, which is the point on the grid where row n and column m intersect.

Note that the Black-Scholes partial differential equation (PDE) is...

$$-r\,f + \frac{\delta f}{\delta t} + r\frac{\delta f}{\delta S_t}S_t + \frac{1}{2}\frac{\delta^2 f}{\delta S_t^2}\sigma^2 S_t^2 = 0 \tag{5}$$

The approximation for the first term in the PDE is...

$$-r\,f \approx -r\,f_{m,n} \tag{6}$$

The backward first order difference approximation for the first derivative of call price with respect to time is...

$$\frac{\delta f}{\delta t} \approx \frac{f_{m,n} - f_{m-1,n}}{\Delta t} \tag{7}$$

The centered first order difference approximation for the first derivative of call price with respect to stock price is...

$$\frac{\delta f}{\delta S_t} \approx \frac{f_{m,n+1} - f_{m,n-1}}{2\,\Delta s} \tag{8}$$

The standard second order difference approximation for the second derivative of call price with respect to stock price is...

$$\frac{\delta^2 f}{\delta S_t^2} \approx \frac{f_{m,n+1} - 2\,f_{m,n} + f_{m,n-1}}{(\Delta s)^2} \tag{9}$$

If we replace the $-r\,f$ in Equation (5) with the approximation in Equation (6), the derivative $\frac{\delta f}{\delta t}$ with the approximation in Equation (7), the derivative $\frac{\delta f}{\delta S_t}$ with the approximation in Equation (8) and the derivative $\frac{\delta^2 f}{\delta S_t^2}$ with the approximation in Equation (9) then the finite difference approximation of the Black-Scholes PDE is...

$$-r\,f_{m,n} + \frac{f_{m,n} - f_{m-1,n}}{\Delta t} + r\frac{f_{m,n+1} - f_{m,n-1}}{2\,\Delta s}S_t + \frac{1}{2}\frac{f_{m,n+1} - 2\,f_{m,n} + f_{m,n-1}}{(\Delta s)^2}\sigma^2 S_t^2 = 0 \tag{10}$$

Noting that stock price at each row n is $n\Delta s$ the finite difference approximation in Equation (10) above becomes...

$$\begin{aligned}-r\,f_{m,n} + \frac{f_{m,n} - f_{m-1,n}}{\Delta t} + r\frac{f_{m,n+1} - f_{m,n-1}}{2\,\Delta s}n\,\Delta s + \frac{1}{2}\frac{f_{m,n+1} - 2\,f_{m,n} + f_{m,n-1}}{(\Delta s)^2}\sigma^2(n\,\Delta s)^2 &= 0\\ -r\,f_{m,n} + \frac{f_{m,n} - f_{m-1,n}}{\Delta t} + r\,n\,\frac{f_{m,n+1} - f_{m,n-1}}{2} + \sigma^2 n^2\frac{f_{m,n+1} - 2\,f_{m,n} + f_{m,n-1}}{2} &= 0\end{aligned} \tag{11}$$

As we march backwards in time we will use the call values at each time m to estimate call value at time $m - 1$. We do this until we get to time zero. Noting that it is $f_{m-1,n}$ that we seek we rewrite the equation above (see Appendix equation (20)) as...

$$f_{m-1,n} = \alpha_1\,f_{m,n-1} + \alpha_2\,f_{m,n} + \alpha_3\,f_{m,n+1} \tag{12}$$

Where...

$$\alpha_1 = \frac{1}{2}(\sigma^2 n - r)\,n\,\Delta t \;\; ...and... \;\; \alpha_2 = 1 - (\sigma^2 n^2 + r)\,\Delta t \;\; ...and... \;\; \alpha_3 = \frac{1}{2}(\sigma^2 n + r)\,n\,\Delta t \tag{13}$$

The Solution To Our Hypothetical Problem

We will divide the time interval to expiration [0,1] into $M = 5$ subintervals per Equation (2) such that...

$$\Delta t = \frac{T}{M} = \frac{1.00}{5} = 0.20 \tag{14}$$

We will define the maximum stock price $S_{max} = \$110$. We will divide S_{max} into $N = 11$ subintervals per Equation (3) such that...

$$\Delta s = \frac{S_{max}}{N} = \frac{\$110}{11} = \$10.00 \tag{15}$$

The table below presents the grid of $N = 11$ rows and $M + 1 = 6$ time interval columns. The grid values for C(1.00) are known as these are the payoffs on the call should stock price get to S(1.00). We apply the finited difference methodology described above to column C(1.00) to get C(0.80). We do this for the remaining columns until a value for C(0.00), which is the call value at time zero, is obtained. The time and space grid looks as follows...

C(0.00)	C(0.20)	C(0.40)	C(0.60)	C(0.80)	C(1.00)	n	S(1.00)
					50.00	11	110.00
				40.60	40.00	10	100.00
			31.19	30.60	30.00	9	90.00
		21.82	21.19	20.60	20.00	8	80.00
	13.00	12.19	11.38	10.60	10.00	7	70.00
5.95	5.10	4.16	3.07	1.74	0.00	6	60.00
	0.96	0.56	0.22	0.00	0.00	5	50.00
		0.02	0.00	0.00	0.00	4	40.00
			0.00	0.00	0.00	3	30.00
				0.00	0.00	2	20.00
					0.00	1	10.00
m = 0	m = 1	m = 2	m = 3	m = 4	m = 5		

Example calculation: Call value C(0.40) row n = 8 of $21.82 is determined as follows...

$$\alpha_1 = \frac{1}{2}(\sigma^2 n - r)\, n\, \Delta t = \frac{1}{2}[(0.20^2)(8) - 0.05](8)(0.20) = 0.216 \tag{16}$$
$$\alpha_2 = 1 - (\sigma^2 n^2 + r)\, \Delta t = 1 - [(0.20^2)(8^2) + 0.05](0.20) = 0.478 \tag{17}$$
$$\alpha_3 = \frac{1}{2}(\sigma^2 n + r)\, n\, \Delta t = \frac{1}{2}[(0.20^2)(8) + 0.05](8)(0.20) = 0.296 \tag{18}$$

Call value per Equation (12) for this example calculation is...

$$f_{2,8} = \alpha_1\, f_{3,7} + \alpha_2\, f_{3,8} + \alpha_3\, f_{3,9} = (0.216)(11.38) + (0.478)(21.19) + (0.296)(31.19) = 21.82 \tag{19}$$

Problem Solution: Call value at time zero per the finite difference method as described above is $5.95. The Black-Scholes value is $6.27. The call value that we calculated via finite differences would have been much closer to the Black-Scholes value had we chosen a value of $M > 5$ and $N > 11$ (i.e. more grid points).

Appendix

A. Equation (11) can be rewritten as...

$$0 = -r\, f_{m,n} + \frac{f_{m,n} - f_{m-1,n}}{\Delta t} + r\, n\, \frac{f_{m,n+1} - f_{m,n-1}}{2} + \sigma^2 n^2 \frac{f_{m,n+1} - 2\, f_{m,n} + f_{m,n-1}}{2}$$
$$\frac{f_{m-1,n}}{\Delta t} = -r\, f_{m,n} + \frac{f_{m,n}}{\Delta t} + \frac{r\, n}{2}\, f_{m,n+1} - \frac{r\, n}{2}\, f_{m,n-1} + \frac{\sigma^2 n^2}{2} f_{m,n+1} - \sigma^2 n^2\, f_{m,n} + \frac{\sigma^2 n^2}{2} f_{m,n-1}$$
$$f_{m,n-1} = -r\, \Delta t f_{m,n} + f_{m,n} + \frac{r\, n\, \Delta t}{2}\, f_{m,n+1} - \frac{r\, n\, \Delta t}{2}\, f_{m,n-1} + \frac{\sigma^2 n^2 \Delta t}{2} f_{m,n+1} - \sigma^2 n^2 \Delta t f_{m,n} + \frac{\sigma^2 n^2 \Delta t}{2} f_{m,n-1}$$
$$f_{m-1,n} = f_{m,n-1}\left\{\frac{\sigma^2 n^2 \Delta t}{2} - \frac{r\, n\, \Delta t}{2}\right\} + f_{m,n}\left\{1 - r\, \Delta t - \sigma^2 n^2 \Delta t\right\} + f_{m,n+1}\left\{\frac{\sigma^2 n^2 \Delta t}{2} + \frac{r\, n\, \Delta t}{2}\right\}$$
$$f_{m-1,n} = f_{m,n-1}\left\{\frac{1}{2}(\sigma^2 n - r)\, n\, \Delta t\right\} + f_{m,n}\left\{1 - (r + \sigma^2 n^2)\, \Delta t\right\} + f_{m,n-1}\left\{\frac{1}{2}(\sigma^2 n + r)\, n\, \Delta t\right\} \tag{20}$$